Year 12 Mathematics

IAS 2.3

Sequences and Series

Robert Lakeland & Carl Nugent

Contents

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NCEA 2 Internal Achievement Standard 2.3 – Sequences and Series

This achievement standard involves applying sequences and series in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply sequences and series in solving problems.	• Apply sequences and series, using relational thinking, in solving problems.	• Apply sequences and series, using extended abstract thinking, in solving problems.

- This achievement standard is derived from Level 7 of The New Zealand Curriculum and is related to the achievement objective
  - use arithmetic and geometric sequences and series

  in the Mathematics strand of the Mathematics and Statistics Learning Area.
- Apply sequences and series in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of concepts and terms
  - communicating using appropriate representations.
- Relational thinking involves one or more of:
  - selecting and carrying out a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts
  - forming and using a model;

  and also relating findings to a context or communicating thinking using appropriate mathematical statements.
- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate a situation
  - identifying relevant concepts in context
  - developing a chain of logical reasoning, or proof
  - forming a generalisation

  and also using correct mathematical statements, or communicating mathematical insight.
- Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.
- Methods include a selection from those related to:
  - the general term of a sequence
  - a partial sum of a sequence
  - the sum to infinity of a geometric series
  - finding the value of the first term, common difference or common ratio of a sequence
  - finding the number of terms in a sequence.
- Methods could require solving equations, which could involve using logarithms.

Sequences

Definitions

A sequence is a succession of numbers separated usually by commas.

Examples could be:

4, 7, 10, 13, . . .

or 8, 16, 32, 64 . . .

Each member of the sequence is called a term and may be represented by the letter t (for term) with a subscript being its position in the sequence.

i.e. t_1 = 1st term

t_2 = 2nd term etc.

Example

Find the first four terms of the arithmetic sequences.

a) $a = 2, d = {}^-5$ b) $a = {}^-4.5, d = 1.2$

a) $2, 2 - 5, 2 - 5 - 5, 2 - 5 - 5 - 5$

$2, {}^-3, {}^-8, {}^-13$

b) ${}^-4.5, {}^-4.5 + 1.2, {}^-4.5 + 1.2 + 1.2, {}^-4.5 + 1.2 + 1.2 + 1.2$

${}^-4.5, {}^-3.3, {}^-2.1, {}^-0.9$

Example

Find the formula for the nth term and 20th term of the arithmetic sequence

$$5, 9, 13, 17, 21, \ldots$$

We know it is arithmetic so using $a = 5$ (first term) and $d = 4$ (common difference) we substitute into the general formula.

$t_n = a + (n - 1)d$

$t_n = 5 + (n - 1)4$ substituting

$t_n = 4n + 1$ simplifying

To find the 20th term we substitute $n = 20$

$t_{20} = 4(20) + 1$

$t_{20} = 81$

Example

Find the number of terms in the arithmetic sequence

$$23, 29, 35, 41, 47, \ldots, 125.$$

Using $a = 23$ (first term) and $d = 6$ (common difference) we substitute into the general formula for an arithmetic sequence.

$t_n = a + (n - 1)d$

$t_n = 23 + (n - 1)6$ substituting

$t_n = 6n + 17$ simplifying

To find the number of terms in the sequence we set the formula for the nth term equal to the last term given in the sequence and solve for n.

$125 = 6n + 17$

$6n = 125 - 17$

$n = 18$

Therefore there are 18 terms in the sequence.

Example

Find the general term for the arithmetic sequence with $a = 4$ and $d = {}^-3$.

$t_n = a + (n - 1)d$

$t_n = 4 + (n - 1){}^-3$

$t_n = {}^-3n + 3 + 4$

$t_n = {}^-3n + 7$

Example

The second term of an arithmetic sequence is 28 and the 6th term 0.

Find the first four terms of the sequence.

Looking at the information we have been given we can represent the sequence as follows.

1st	2nd	3rd	4th	5th	6th	...
a,	28,	28 + d,	28 + 2d,	28 + 3d,	28 + 4d,	...

Yet the 6th term = 0 so

$T_6 = 0$ gives $28 + 4d = 0$

$d = {}^-7$

Constant difference = ${}^-7$ so the first 4 terms

1st	2nd	3rd	4th	and	5th	6th	...
35,	28,	21,	14,		7,	0,	...

Example

An arithmetic sequence has 5th term of 24 and 7th term of 40. What are the first three terms and the rule for the sequence?

Between the 5th and 7th terms we have two constant differences so

$2d = 40 - 24$

$d = 8$

Therefore $t_n = a + (n - 1)8$

We know the 5th term is 24 therefore substituting this in the rule above we get

$24 = a + (5 - 1)8$

$24 = a + 32$

$a = {}^-8$

Therefore $t_n = {}^-8 + (n - 1)8$

$= 8n - 16$

Substituting $n = 1, 2$ and 3 for the first three terms gives $t_1 = {}^-8$, $t_2 = 0$ and $t_3 = 8$.

Applications of Arithmetic Sequences

Applications of Arithmetic Sequences

In this section we look to apply our knowledge of arithmetic sequences to real problems in common situations.

Like any application or word type problem, it is important to identify what is actually being asked and to use the appropriate formula(e).

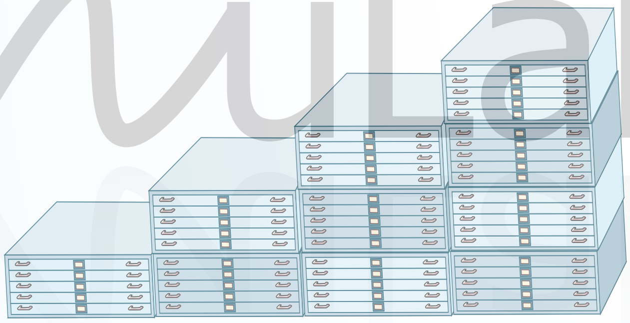
Sometimes drawing a diagram or just listing down the sequence of numbers that are given, can help to clarify the problem.

With application problems that involve arithmetic sequences it is important to identify the key parameters of the sequence such as the first term and common difference before proceeding.

Extra Notes

Example

A company sells a storage stacking system comprising cabinets, that can fit on top of one another. Each cabinet contains 5 sliding drawers.



a) A customer has decided to use the same arrangement to that depicted in the diagram (i.e. column 1 has one cabinet of 5 drawers, then column 2 has two cabinets each of 5 drawers etc.)

How many drawers in total would she have with a 13 column arrangement?

b) If the same customer as in a) purchased an additional 65 cabinets, how many more **completed** columns does it add to her existing arrangement?

Solution

a) Sequence of drawers is 5, 10, 15, 20, ...

Using $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{13} = \frac{13}{2}[2(5) + (13-1)5]$

$S_{13} = 455$ drawers

b) The next column would hold 14 cabinets and the one after that 15 etc.

Since 14 + 15 + 16 + 17 = 62,

therefore it would add an additional four columns.

Excellence – Investigate solutions to the following problems.

86. Ann has been saving for her retirement. She has saved $89 640 after 24 years.

Each year she has increased her savings from the previous year by the same amount. In Year 10 she saved $3210. How much money did Ann save in the first year of her savings plan?

87. A company manufacturing a plastic cover or skin for the iPhone knows that producing 500 covers will cost $9 650 or $19.30 each. To produce 1250 covers will cost $15 837.50 or $12.67 each.

Market research has shown that the company can only sell the covers at $15 each.

There are both fixed costs and variable costs when producing the covers.
The fixed cost is the same no matter how many covers are produced. The variable cost depends directly on the number of covers produced.

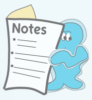
Using these figures, what is the minimum number of covers that must be sold in order to make a profit?

116. Joe is now paid \$16.00 an hour. How much was he paid per hour when he was first employed?

117. Find the formula that relates the wage per hour that Joe is paid to the number of years he has been working for the company.

118. Joe plans to retire when his wage reaches \$30 per hour. Joe joined the company when he was 15 years old. How old will he be when he plans to retire?

Sums of Geometric Sequences



Partial Sums of Geometric Sequences

A partial sum is the sum of a fixed number or finite number of terms of a sequence.

Consider the geometric sequence 3, 6, 12, 24, 48, ...

Now $S_1 = 3$

$S_2 = 3 + 6 = 9$

$S_3 = 3 + 6 + 12 = 21$ etc.

The letter S denotes a partial sum and the subscript identifies the number of terms being added.

The partial sum S_8 of the sequence above is

$$S_8 = 3 + 6 + 12 + 24 + 48 + 96 + 192 + 384 \quad ①$$

A long sequence like this would be cumbersome to add up so we look for a shorter method.

If we now multiply the partial sum S_8 by the common ratio of the sequence (i.e. 2) we obtain

$$2S_8 = 6 + 12 + 24 + 48 + 96 + 192 + 384 + 768 \quad ②$$

Seven members of this new sequence were in the old equation ①.

If we now subtract equation ① from equation ② we obtain

$$2S_8 = \quad 6 + 12 + 24 + 48 + 96 + 192 + 384 + 768 \quad ②$$

$$S_8 = 3 + 6 + 12 + 24 + 48 + 96 + 192 + 384 \quad ①$$

$$S_8 = {}^{-}3 + 768 \quad ② - ①$$

so $S_8 = 765$ (the partial sum of 8 terms)

We can now use a similar approach to obtain a general formula to find the sum of n terms of a geometric sequence.

Consider the partial sum S_n of a geometric sequence.

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} \quad ①$$

Multiplying the partial sum by r the common ratio we obtain

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n \quad ②$$

Subtracting equation ① from equation ② we obtain

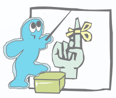
$$rS_n = \quad ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n \quad ②$$

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} \quad ①$$

$$rS_n - S_n = ar^n - a \quad ② - ①$$

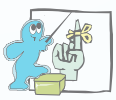
$$S_n(r - 1) = a(r^n - 1) \quad \text{Factorising}$$

$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$



We obtain a series by summing the terms of a sequence. A series may be finite (i.e. a partial sum) or infinite.

The sum of n terms of a series is often called the nth partial sum.



The formula $S_n = \frac{a(r^n - 1)}{(r - 1)}$ enables us to find the sum of n terms of a geometric sequence.

It is sometimes also written in the form

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

Make sure you are familiar with at least one of these forms.

Applications of Geometric Sequences

Applications of Geometric Sequences

In this section we look to apply our knowledge of geometric sequences in applications and common situations.

Like any application or word type problem it is important to identify what is actually being asked and to use the correct formula or combination of formulae, i.e. partial sum, sum to infinity or the formula for the general term to get the answer.

Sometimes drawing a diagram can help to clarify the problem or just listing down the sequence of numbers that are given can provide the catalyst to solving it.

Extra Notes

Example

A person who initially weighs 120.0 kg begins a diet. The first month they lose 18.0 kg, the next month 12.0 kg and the following month 8.0 kg and they continue to lose weight by decreasing by the same ratio each successive month.

a) Give a formula that represent, the amount of weight lost in a particular month n and find out how much weight the person would be expected to lose in month 10.

b) How much would the person weigh after 10 months on the diet?

c) If the person continued on the diet indefinitely what would be the maximum amount of weight they could lose and their final weight?

Solution

a) Formula for the amount of weight lost in a particular month n is

$$T_n = 18 \times \left(\frac{2}{3}\right)^{n-1}$$

In month 10 the expected weight lost is

$$T_{10} = 18 \times \left(\frac{2}{3}\right)^{9}$$

$$T_{10} = 0.468 \text{ kg} \quad (3 \text{ sf})$$

b) To find the amount a person would weigh after 10 months on the diet we need to find the total weight lost and subtract this amount from the person's original weight.

$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$

$$S_{10} = \frac{18\left(\left(\frac{2}{3}\right)^{10} - 1\right)}{\left(\left(\frac{2}{3}\right) - 1\right)}$$

$$S_{10} = 53.1 \text{ kg} \ \ (1 \text{ dp})$$

Weight of person after 10 months is

$$120.0 - 53.1 = 66.9 \text{ kg} \ \ (1 \text{ dp})$$

c) If the person continued on the diet indefinitely the maximum weight loss can be calculated using the sum to infinity formula.

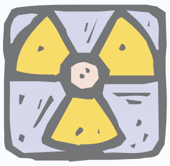
$$S_\infty = \frac{18}{\left(1 - \frac{2}{3}\right)}$$

$$S_\infty = 54 \text{ kg}$$

Final weight $= 120 - 54$

$= 66$ kg

Example



The radioactive element caesium has a half-life of 30.1 years and we initially start with 100 grams

a) what is the rate of decay?

b) how much of the substance is left after 12 years?

c) how long will it take to reduce to 5 grams of radioactive caesium?

We use the decay formula with $D_n = 50$, $O = 100$ and $n = 30.1$, because we know that in 30.1 years only half of the original amount (100 g) of the substance will exist i.e. 50 grams.

a)

$$D_n = O \times \left(1 - \tfrac{r}{100}\right)^n$$

$$50 = 100 \times \left(1 - \tfrac{r}{100}\right)^{30.1}$$

$$0.5 = \left(1 - \tfrac{r}{100}\right)^{30.1}$$

We now take the 30.1th root of each side.

$$0.977\,234\,978\,6 = \left(1 - \tfrac{r}{100}\right)$$

$$r = 2.276\,502\,14$$

$$r = 2.3\%$$

So the substance is decaying at the rate of 2.3% pa.

b) To find the amount of the substance left after 12 years we substitute the appropriate values for O, r and n into the decay formula.

$$D_n = O \times \left(1 - \tfrac{r}{100}\right)^n$$

$$D_n = 100 \times \left(1 - \tfrac{2.3}{100}\right)^{12}$$

$$D_n = 76 \text{ g} \quad \text{(2 sf)}$$

After 12 years 76 grams of the caesium remain.

c) To find the time to reduce the radioactive caesium to 5 g we substitute the appropriate values for D, O, and r into the decay formula.

$$D_n = O \times \left(1 - \tfrac{r}{100}\right)^n$$

$$5 = 100 \times \left(1 - \tfrac{2.3}{100}\right)^n$$

$$n = 128.746$$

$$n = 130 \text{ years (2 sf)}$$

If we use the Solver on our graphics calculator we can put in the variables we know and solve for the unknown variable.

Using your TI-84 Plus

To solve any equation on the TI it should be in the form

$$0 = \text{Equation}$$

so in this case we will enter the equation in the form

$$0 = D_n - O \times \left(1 - \tfrac{r}{100}\right)^n$$

Go to the Solver by pressing [MATH] and scrolling down.

If there is already an equation entered you will need to delete it.

Hit the up arrow key and [CLEAR] to erase the function that was previously there. Then enter the equation to solve.

Up

[▲] [CLEAR] [ALPHA] [D] [–] [ALPHA] [O]

[(] [1] [–] [ALPHA] [R] [÷] [1]

[0] [0] [)] [^] [ALPHA] [N] [ENTER]

It now displays your equation at the top and each variable.

D – O(1 – R/100) ^ N = 0
D = 50
O = 100
R = ▮
N = 30.1

Using the cursor keys to enter in D, O and N and place the cursor next to R and select [ALPHA] [ENTER] to solve.

Using your Casio 9750GII

Go to the Solver with [MENU] followed by [8] [F3] to select the Solver.

Enter the equation

[ALPHA] [D] [SHIFT] [=] [ALPHA] [O] [(]

[1] [–] [ALPHA] [R] [÷] [1] [0]

[0] [)] [^] [ALPHA] [N] [EXE]

Again it will list the variables and you enter all the known variables and place the cursor next to the unknown (in this case R) and select solve [F6].

Using Spreadsheets for Sequences and Series

Using a Spreadsheet

We can use a spreadsheet for sequences to quickly calculate values and to add up terms of a series. A big advantage of spreadsheets is you can have a printed record of all the results which is useful if you are investigating different sequences and need to compare terms.

The instructions given here are for Microsoft Excel but apply similarly to Open Office and / or Mac Numbers.

A spreadsheet is an array of cells which can have a formulae entered into them. For example, if we had an arithmetic sequence we would head

n	Term	Sum
1	23	23
2	31	54
3	39	93
4	47	140
5	55	195
6	63	258

the three coumns with their labels and put the term number in column 1. Then we could either set the first term to 23 and the formula in the cell directly underneath to be

Cell = Cell above + 8

This cell could then be copied down the column. For the sums we can then add up the total of all the terms in the series up to the cell we are at.

This is explained in the first **Example.**

Alternatively we can enter the formula for the general term of an arithmetic sequence.

Cell = 23 + (Cell to the left – 1) x 8.

Example

Use a spreadsheet to find the 15th term and the sum of the first 15 terms of the sequence 23, 31, 39, 47, 55, ...

Open a blank spreadsheet and put the labels n, Term and Sum as labels in the first three columns.

	A	B	C
1	n	Term	Sum

Starting in cell A2 and going down enter the term number 1, 2, 3, and 4. Now with your mouse click in cell A2 and sweep to cell A5. Release the mouse button and move the mouse to the bottom right corner. The mouse icon will change to cross hairs.

	A
1	n
2	1
3	2
4	3
5	4

Select this corner and drag it down the spreadsheet until you have sufficient rows (n = 15). Now enter 23 in cell B2 and in cell B3 enter '= B2 + 8'. Instead of typing B2 it is possible to click on the cell B2 to enter it.

	A	B	C
1	n	Term	Sum
2	1	23	
3	2	=B2+8	

Now click once on this cell B3 and release the mouse button and again select the bottom right corner and drag this one cell down to the row with n = 15 in it.

Solution cont...

In the cell C2 enter =sum(B$2:B2).

The B$2 refers to the starting cell and the $ instructs the spreadsheet not to change this.

B	C
Term	Sum
23	=sum(B$2:B2)
31	

After entering this formula click on the cell and select the bottom right corner and drag this one cell down to the row with n = 15 in it.

You should now have the first 15 terms of the sequence and the first 15 partial sums.

	A	B	C
1	n	Term	Sum
2	1	23	23
3	2	31	54
4	3	39	93
5	4	47	140
6	5	55	195
7	6	63	258
8	7	71	329
9	8	79	408
10	9	87	495
11	10	95	590
12	11	103	693
13	12	111	804
14	13	119	923
15	14	127	1050
16	15	135	1185

Term 15 is 135 and the sum of the first 15 terms is 1185.

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102. $157\,837\,977 = 11 \times 3^{n-1}$
$n = 16$

103. $n = 7$

104. $n = 8$

105. $n = 6$

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106. $7r^3 = 189$

$r = 3$ Seq. $\langle 7, 21, 63, \ldots \rangle$

107. $256r^3 = 2048$

$r = 2$ Seq. $\langle 64, 128, 256, \ldots \rangle$

108. $r = \pm\frac{1}{4}$ (± 0.25) and
$a = \pm 81\,920$

Seq. $\langle {}^{-}81\,920, {}^{-}20\,480, {}^{-}5120, \ldots \rangle$
Seq. $\langle {}^{-}81\,920, 20\,480, {}^{-}5120, \ldots \rangle$

109. $r = \frac{1}{9}$ Seq. $\langle 129\,140\,163, 14\,348\,907, 1\,594\,323, \ldots \rangle$

110. $r = \frac{2}{3}$ $(0.66\dot{6})$

Seq. $\langle 9, 6, 4, \ldots \rangle$

111. $r = ab$

Seq. $\langle a, a^2b, a^3b^2, \ldots \rangle$

112. $t_{12} = 4.00 \times (1.15)^{12-1}$

$t_{12} = 18.61$ km

113. $t_{20} = 21.0 \times (1.18)^{20-1}$

$t_{20} = 487.5$ g

114. $t_n = 850 \times (0.95)^{n-1}$

$100 = 850 \times (0.95)^{n-1}$

$n - 1 = 41.7,\ n = 42.7$
Answer = 43.

115. If the wage increases by 3% per year then the hourly rate must increase by 3%. A 3% increase is the same as multiplying by a common ratio of 1.03.

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116. Over 21 years ago means he has had 20 increases at the constant ratio of 1.03.

$t_1 = 16.00 \div (1.03)^{20}$

$t_1 = \$8.86$ per hour

Page 23 cont...

117. $t_n = 8.86 \times (1.03)^{n-1}$

118. $30 \le 8.86 \times (1.03)^{n-1}$

$n \ge 42.3$

$n = 43$

Started work at 15 so he will retire at 58 years.

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119. $S_{10} = 5115$

120. $S_{16} = {}^{-}109\,225$

121. $S_8 = 1\frac{127}{128}$ $(1.992\,187\,5)$

122. $S_{20} = 67.50$ (4 sf)

123. $S_{16} = 5247$ (4 sf)

124. $S_{18} = 13.33$ (2 dp)

125. $S_{10} = 4.50$ (2 dp)

126. $S_{13} = 8.27$ (2 dp)

127. $S_{5001} = 1$

128. $n = 13$

$S_{13} = 2730\frac{1}{3}$ $(2730.\dot{3})$

129. $n = 7$

$S_7 = 64.00$ (2 dp)

130. $n = 8$

$S_8 = 1.275$

131. $S_5 = 15\,620$ words

132. $S_{10} = \$1\,432.91$

133. $S_{20} = 1\,373$ (0 dp) minutes

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134. $a = \$28\,656$

$S_{15} = \$642\,000$ (3 sf)

135. $a = 1320$

$t_{10} = 2639$

136. $t_n = 20 \times (1.1)^{n-1}$

$n = 11$

$S_{11} = 371$ km

137. $r = 0.794$

$a = 5.04$ kg

$S_8 = 20.6$ kg (3 sf)

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138. $S_\infty = 20$

139. $S_\infty = 6$

140. $S_\infty = 50$

Page 28 cont...

141. $S_\infty = \dfrac{189}{\left(1 - \frac{1}{3}\right)}$

$S_\infty = 283.5$

142. $S_\infty = 0.4$

143. $S_\infty = 14.29$ (2 dp)

144. $S_\infty = 56.89$ (2 dp)

145. $S_\infty = 0.\dot{3}\left(\frac{1}{3}\right)$

146. $S_\infty = 347.\dot{2}\left(347\frac{2}{9}\right)$

147. $r = \frac{2}{3}$, Seq. $\langle 9, 6, 4, \frac{8}{3}, \ldots \rangle$

148. $r = \frac{{}^{-}5}{7}$,

Seq. $\langle 14, {}^{-}10, 7\frac{1}{7}, {}^{-}5\frac{5}{49}, \ldots \rangle$

$\langle 14, {}^{-}10, 7.14, {}^{-}5.10, \ldots \rangle$

149. $\langle 28.8, {}^{-}5.76, 1.152, {}^{-}0.2304, \ldots \rangle$

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150. $r = 0.4$, $a = 125$, $S_\infty = 208.\dot{3}$

151. $r = \frac{1}{4}$, $a = 16\,384$, $S_\infty = 21\,845.\dot{3}$

152. $r^2 = \frac{4}{9}$, $r = \frac{\pm 2}{3}$, $a = \pm 54$

when $r = \frac{2}{3}$, $S_\infty = 162$

when $r = \frac{{}^{-}2}{3}$, $S_\infty = {}^{-}32.4$

153. $r^6 = \frac{1}{4096}$, $r = \frac{\pm 1}{4}$, $a = 64$

when $r = \frac{1}{4}$, $S_\infty = 85.\dot{3}$

when $r = \frac{{}^{-}1}{4}$, $S_\infty = 51.2$

154. $a + ar = 40$

$a(1 + r) = 40$

$$72 = \frac{a}{(1 - r)}$$

$$72 = \frac{40}{(1 + r)(1 - r)}$$

$$1 - r^2 = \frac{40}{72}$$

$$r^2 = \frac{4}{9}$$

$$r = \frac{\pm 2}{3}$$

when $r = \frac{2}{3}$, $a = 24$

when $r = \frac{{}^{-}2}{3}$, $a = 120$